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THE FOUNDATIONS OF TRIGONOMETRY

By Arthur C. Lunn

Introduction. In the existing literature of real analysis the purely logical introduction of the trigonometric functions, when done on the basis of theorems of analysis alone without appeal to any geometric setting to supply features of the proofs, seems to be accomplished essentially in two ways, each of which is interpretable in terms of either real or complex variables.

According to the first method the functions sine and cosine, in terms of which the others are defined, are introduced by definition in terms of the series

$$(1)\quad S(x) = x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \frac{x^7}{\underline{|7}} + \cdots, \qquad C(x) = 1 - \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} - \frac{x^6}{\underline{|6}} + \cdots.$$

Then the proof of the permanent convergence of these series shows the existence of the functions for every value of the argument, and the theory of infinite series furnishes the basis of the proof that they possess the properties ascribed by elementary trigonometry to the sine and cosine of an angle whose radian measure is x.

According to the second method these functions are distinguished as solution of the differential equations

$$(2)\qquad S'(x) = C(x), \qquad C'(x) = -\,S(x),$$

with the special conditions

$$S(0) = 0, \qquad C(0) = 1.$$

These four conditions may then be viewed as a set of postulates, easily shown to be independent, involving as undefined elements the functions S and C, whose existence and unique determination are shown by means of the theory of differential equations.

But these two methods are hardly as elementary as might be desired, implying as they do a developed theory, in the one case of infinite series, in the other of differential equations. Moreover, the connection with the intuitional origin of the functions out of geometric constructions is so remote, that it is not easy to decide what properties of these functions must be proved in the critical theory in order to identify them completely as the analytic equivalents of the sine and cosine as understood in the practical trigonometry of lines and angles. The object here is to offer a supplementary point of view, admitting logical development of equal rigor, but based more directly on general theorems only of analysis, and more directly interpretable in terms of elementary trigonometric notions.

Part I gives a set of postulates chosen suitably from familiar trigonometric propositions, with proof that they determine uniquely the functions sine and cosine and the constant π. Part II shows how the known expressions for sine, cosine, and angle, in terms of the tangent of the half-angle as auxiliary, may be regarded as definitions which lend themselves readily to a proof of the existence of functions satisfying these postulates. Part III contains a proof of the independence of the postulates, admitting the logical consistency of the preceding theory. Attention is confined to the case of real variables.

I. Postulates and Proof of Uniqueness. Let $S(x)$ and $C(x)$ be symbols of two undefined single-valued functions of the real variable x, and p the symbol of an undefined constant. With the admission, as part of the logical background, of such propositions of real analysis as will be specified, directly or by implication, it will be shown that these functions and this constant are uniquely determined by the following postulates, which are thus sufficient, not only to serve as basis for the development of analytic trigonometry, but also in the sense that their justification by inference from diagrams would make the connection with geometric trigonometry complete. For brevity, P will be used to represent the set of all numbers of the form $ip/2^j$, where i, j are integers.

POSTULATES.

1. $S(x+y) = S(x)C(y) + C(x)S(y)$, *if* x, y *are numbers of the set* P.
2. $S(x)$ *is continuous for every* x.
3. $C(x)$ *is continuous for every* x.
4. $C(p/2) = 0$.
5. $S(p/2) = 1$.
6. $C(p) = -1$.
7. $C(x)$ *is not negative for any* x *of the set* P *on the interval* $0 \cdots p/2$.
8. *If* $S(x)/x$ *has a limit as* x *approaches zero over the set of numbers* $p/2^n$, *n being a positive integer, that limit is unity.*

The first postulate is the addition formula for the sine, stated, however, only for angles which can be constructed from the angle p (straight angle) by successive bisections and additions; the seventh may be said to identify $p/2$ as the angular measure of a quadrant, whatever be the value of p; while the eighth, suggested by the construction of the sequence of inscribed polygons of 2^n sides, is needed to determine the number p and thus fix the unit of measure of the angles. The first seven postulates alone determine the functions $\sin(\pi x/p)$, $\cos(\pi x/p)$, leaving p arbitrary.

Since the value of a continuous function at any point is determined by its values for the points of any set everywhere dense, it will be sufficient to prove that the values of S and C are determined for every point of P, which is such a set. For the following proof of uniqueness then, unless otherwise stated, x, y will be understood to belong to P.

From postulates 1, 4, 5, we have

$$S(x+p/2) = C(x), \qquad S(p) = 0, \tag{3}$$

so that

$$C(x+p/2) = S(x+p),$$

which by the aid also of 6 becomes

$$C(x+p/2) = -\,S(x)\,; \tag{4}$$

hence, by substitution of $x - p/2$ for x,

$$S(x-p/2) = -\,C(x), \qquad C(x-p/2) = S(x). \tag{5}$$

These equations, with arguments on the two sides differing by $p/2$, give by induction all of the familiar formulae needed for "reduction to the first quadrant," and also by substitution of $x + y$ for x the addition formula for the cosine:

$$C(x+y) = S(x+y+p/2) = C(x)C(y) - S(x)S(y). \tag{6}$$

The values of S and C will therefore be determined everywhere by their values at the points of the set P_1, defined as the points of P which lie on the interval $0 \cdots p/2$, or "first quadrant." It may be noted that on account of the continuity assumed, all of the preceding formulae are immediately proved valid for any values of x and y.

Now let

$$\sigma(x) = \overline{S(x)}^2 + \overline{C(x)}^2\,;$$

then

$$\sigma(x+y) = \sigma(x)\,\sigma(y), \qquad \sigma(2x) = \overline{\sigma(x)}^2\,;$$

and since $\sigma(p) = 1$, the latter gives by induction $\sigma(p/2^n) = 1$, where n is any positive integer, there being no ambiguity of sign in the square root because σ is a sum of squares. Hence, by the addition formula for σ just given, and by continuity, $\sigma(x) = 1$ for every x; that is,

$$\overline{S(x)}^2 + \overline{C(x)}^2 = 1.$$

On this account comes in a familiar way

$$C(2x) = \overline{C(x)}^2 - \overline{S(x)}^2 = 2\,\overline{C(x)}^2 - 1,$$

giving

$$\overline{C(x)}^2 = \frac{1 + C(2x)}{2}, \tag{7}$$

which, starting at $x = p/2$, determines C successively for all points of the set $p/2^n$, the ambiguity of sign being removed by postulate 7. Then S is determined for the same set by

$$S(x) = \frac{S(2x)}{2\,C(x)}, \tag{8}$$

where, as appears from the preceding determination of $C(x)$, the denominator is not zero for any value of x concerned here. Then the addition formulae complete the determination of S and C for all points of the set P_1.

To show that the constant p is determined, let

$$x_i = \frac{p}{2^{i+1}}, \qquad S_i = S(x_i), \qquad C_i = C(x_i), \qquad L_i = \frac{S_i}{x_i}.$$

Then the recursion formulae $x_{i-1} = 2x_i$, $S_{i-1} = 2S_iC_i$ give $L_{i-1} = L_iC_i$, in which $L_0 = 2/p$; so that

$$\frac{p}{2} L_n = M_n = \prod_{i=1}^{n} \frac{1}{C_i}.$$

Here the numbers M_n, $n = 1, 2, 3, \cdots \infty$, form a monotonic increasing sequence, since every C_i is positive and less than unity. But the sequence is bounded: for

$$C_i^2 = \frac{1 + C_{i-1}}{2}, \qquad C_0 = 0,$$

which gives successively

$$\begin{array}{ll} C_1^2 = \frac{1}{2}, & C_1 > \frac{1}{2}; \\ C_2^2 > \frac{3}{4}, & C_2 > \frac{3}{4}; \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ C_i^2 > 1 - \frac{1}{2^i}, & C_i > 1 - \frac{1}{2^i}; \end{array}$$

and therefore

$$M_n < \frac{1}{\left(1-\frac{1}{2}\right)\left(1-\frac{1}{4}\right)\cdots\left(1-\frac{1}{2^n}\right)} \leqq \frac{1}{\frac{1}{2}\left[1-\frac{1}{4}-\frac{1}{8}-\cdots-\frac{1}{2^n}\right]} \leqq 4.$$

By a fundamental proposition due to Weierstrass this sequence has therefore a definite finite limit; hence the sequence of values of L_n has a limit, which by postulate 8 is unity; and hence $p = \pi$, the definite number defined by

$$\frac{\pi}{2} = \lim_{n=\infty} M_n, \tag{9}$$

where

$$M_1 = \frac{1}{\sqrt{\frac{1}{2}}}, \qquad M_2 = \frac{1}{\sqrt{\frac{1}{2}}} \cdot \frac{1}{\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}},$$

$$M_3 = \frac{1}{\sqrt{\frac{1}{2}}} \cdot \frac{1}{\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}} \cdot \frac{1}{\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}}, \cdots.^{*}$$

The proof of the uniqueness of the system determined by postulates 1–8 is thus complete.

The particular set of postulates here used is of course arbitrary, and subject to endless possible modifications without loss of its desired character. One change perhaps desirable would be to state the addition theorem or functional equation only for acute angles, and to supplement it by such conditions as to make the values of the functions in the other quadrants determinate. Another method in the same spirit as the above would be to state a set of postulates involving a single undefined function such as to determine it as the tangent of the angle, and then to define the sine and cosine in terms of it, or preferably in terms of the tangent of the half-angle so as to avoid irrational expressions.

II. Existence Proof. For the further development of the consequences of the above postulates the proof could run mainly along conventional lines, leading among other things to the equations (2) for the derivatives, and from these by Taylor's theorem to the series in (1). For the present purpose it may be noted that if t be the tangent of the half-angle, defined by

$$t = S(x/2)/C(x/2),$$

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* Geometrically, M_n is one-fourth the perimeter of a regular polygon of 2^{n+1} sides, inscribed in a circle of unit radius.

there would emerge the formulae

$$(10)\qquad S(x) = \frac{2t}{1+t^2}, \quad C(x) = \frac{1-t^2}{1+t^2},$$

while the angle x would be related to t by a differential equation integrable by quadrature:

$$\frac{dt}{dx} = \frac{1+t^2}{2}.$$

These equations suggest a simple means of proving the existence of functions which satisfy the postulates of part I. For, let

$$(11)\qquad I(v, w) = \int_v^w \frac{dz}{1+z^2},$$

and let the number π be defined by

$$(12)\qquad \pi = 2\, I(0, \infty),$$

which is finite; then by the Archimedean property of the linear continuum, any value x of the fundamental independent variable (angle) can be written in the form

$$(13)\qquad x = \xi + 2r\pi,$$

where r is an integer, and ξ is a number, uniquely determined, satisfying the condition $-\pi < \xi \leqq \pi$. Moreover, since the function under the integral sign is always positive, the integral $I(0, t)$ is a monotonic continuous function of its upper limit t and ranges from $-\pi/2$ to $+\pi/2$ as t ranges from $-\infty$ to $+\infty$. The equation

$$(14)\qquad \xi = 2I(0, t)$$

for any ξ numerically less than π thus determines t uniquely as a monotonic continuous function of ξ, and therefore of x for all values between any two successive odd multiples of π.

Equations (10) may then be used as indirect definitions of functions $S(x)$ and $C(x)$, making these exist as single-valued continuous functions of x (postulates 2 and 3) for all values of x except those of the form $(2r+1)\pi$. Continuity at these points and also satisfaction of postulate 6 are secured by interpreting the definition so as to imply also the limiting values $S = 0$, $C = -1$ for $t = \pm\infty$; for these limits are independent of the sign of t as its numerical value approaches infinity, this sign depending on whether x is above or below the "straight-angle" value concerned.

Now the substitution $z = 1/z'$ transforms the integral $I(0, 1)$ into $I(1, \infty)$, so that $\pi/2 = 2I(0, 1)$ since $\pi = 2\,\{I(0, 1) + I(1, \infty)\}$. This gives the related values: $x = \pi/2$, $t = 1$, and for these $S = 1$, $C = 0$, as in postulates 4 and 5. Moreover, as x ranges over the interval $0 \cdots \pi/2$, t ranges over the interval $0 \cdots 1$, hence C is not negative there (postulate 7).

If x is numerically less than π, the mean-value theorem applied to the integral $I(0, t)$ shows that x lies between $2t$ and $2t/(1 + t^2)$, so that $S(x)/x$ lies between $1/(1 + t^2)$ and 1; as x approaches 0, t must approach 0, therefore

$$\lim_{x=0} \frac{S(x)}{x} = 1,$$

a stronger form of postulate 8.

On account of continuity it is sufficient to verify the addition formula for angles x, y which are not integral multiples of π, so that they may be written

$$x = 2I(0, t) + 2r\pi, \qquad y = 2I(0, u) + 2s\pi, \tag{15}$$

where t, u, are finite. Now the substitution

$$z = \frac{z' - t}{1 + z't}, \qquad z' = \frac{z + t}{1 - zt}, \qquad \frac{dz}{1 + z^2} = \frac{dz'}{1 + z'^2}$$

transforms the integrand function of $I(0, u)$ into itself; but with respect to the limits of integration a distinction must be made depending on whether the point of discontinuity $z = 1/t$ occurs on the interval of integration; we have then

$$I(0, u) = \begin{cases} I(t, v) & \text{when } tu < 1, \\ I(t, \infty) & \text{when } tu = 1, \\ I(t, \infty) + I(-\infty, v) & \text{when } tu > 1, \end{cases}$$

where

$$v = \frac{t + u}{1 - tu}, \tag{16}$$

and corresponding to these respectively

$$x + y = \begin{cases} 2\,I(0, v) + 2(r + s)\,\pi, \\ (2r + 2s + 1)\,\pi, \\ 2\,I(0, v) + 2(r + s + 1)\,\pi. \end{cases} \tag{17}$$

The intermediate case where $tu = 1$ may be treated by means of limits, while the other two show that if $tu \neq 1$, then v as defined is that value of the parameter in the integral which corresponds to the sum of the angles. Hence

$$(18) \qquad S(x+y) = \frac{2 \cdot \dfrac{t+u}{1-tu}}{1 + \left(\dfrac{t+u}{1-tu}\right)^2} = \frac{2t(1-u^2) + 2u(1-t^2)}{(1+t^2)(1+u^2)},$$

or

$$S(x+y) = S(x)\,C(y) + C(x)\,S(y),$$

as required by postulate 1. This completes the verification of the existence of functions satisfying the postulates.

As an alternative method, the entire trigonometric theory could be based directly on the definitions contained in equations (10), (13), (14), and would be equivalent to any theory containing postulates 1–8 as propositions, since they were shown to be sufficient for unique determination of the undefined elements.

III. Proof of Independence. Let it now be granted that there exist in the realm of real analysis functions having the properties of the trigonometric sine and cosine, and also, for brevity, that a certain amount of the theory of the exponential function, which occurs in two of the examples to be given, has been deduced in the usual manner; then it remains to be shown that the postulates of part I are independent. According to a recognized method this is done by the following series of examples, each of which gives a possible interpretation for the symbols S, C, p, such as to violate the postulate corresponding to the number annexed while fulfilling all of the others. Details of verification need not for the most part be set down.

1. $S(x) = x\left(1 - \dfrac{2p-4}{p^2}x\right)$; $C(x) = 1 - \dfrac{2x}{p}$; p arbitrary. Here $S(x+y)$ and $S(x)C(y) + C(x)S(y)$ differ by $\dfrac{4xy}{p^2}\left\{2 - \dfrac{p-2}{p}(x+y)\right\}$, so that postulate 1 is violated; all the other postulates are satisfied.

2. $S(x) = \sin x$ for points of the set P, but having arbitrary values differing from those of $\sin x$ for other values of x; $C(x) = \cos x$ for every x; $p = \pi$. Note here that all the postulates aside from those of continuity refer only to the points of the set P.

3. Similar to 2, except that it is $C(x)$ which takes arbitrary values, differing from those of $\cos x$, for values of x outside of the set P.

4. $S(x) = a^x \dfrac{\sin 5x}{5}$; $C(x) = a^x \cos 5x$; $p =$ the smallest positive number satisfying the equation $\sin (5p/2) = 5/7$ (approximately $p = 0.318$); and $a = 7^{\frac{2}{p}}$. Here $C(p/2) = \sqrt{24}$.

5. $S(x) = \dfrac{\sin 2x}{2}$; $C(x) = \cos 2x$; $p = \dfrac{\pi}{2}$. Here $S(p/2) = 1/2$. The postulate concerned may be said to fix the vertical scale of the sine curve.

6. $S(x) = a^x \dfrac{\sin 2x}{2}$; $C(x) = a^x \cos 2x$; $p = \dfrac{\pi}{2}$; and $a = 2^{\frac{2}{p}}$. This makes $C(p) = -4$.

7. $S(x) = \sin x$; $C(x) = \cos x$; $p = 5\pi$. This brings positive and negative quarter-waves into the interval $0 \cdots p/2$ for the curve of $C(x)$.

8. $S(x) = \sin mx$; $C(x) = \cos mx$; $p = \dfrac{\pi}{m}$; where $m \neq 1$. Here $\lim\limits_{x=0} \dfrac{S(x)}{x} = m$. For example, m might be $\dfrac{\pi}{180}$, so that x would be measured in degrees.

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